STABILITY ANALYSIS OF λ–MODEL FOR HUMAN MOTOR SYSTEM WITH TINY MOMENT OF INERTIA

Lan Li[1], Zhu Kuanyi[1]

[1] School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore, Singapore

e-mail: lanli@pmail.ntu.edu.sg

Abstract-**This short paper studies the properties of λ-model, which is a human motor system model derived from equilibrium point hypothesis. The stability of the λ-model based on equilibrium and Jacobian matrix is investigated, and some mathematical and simulation results are presented. Especially, the behavior of the λ-model with tiny moment of inertia is discussed. The results show that the λ-model is stable and has a unique equilibrium point under the normal condition. But when the moment of inertia is tiny, the stability and equilibrium point will be determined by the values of the moment of inertia and initial points.**

Keywords **- Human motor system, λ-model, Stability, Equilibrium point, Moment of inertia.**

I. INTRODUCTION

Motor disability is the one of the worst aftereffects of the neuro-musculo-skeletal system injury. Patients often have to depend on other persons or devices for even simple tasks. The purpose of medical management and rehabilitation engineering is to provide independence and mobility to a disabled person, which will ultimately improve his quality of life. However, the outstanding rehabilitation technology is built on the comprehensive understanding about human motor system.

In this short paper, the stability of λ-model is discussed, which will be useful for deeper understanding of human motor system. Especially, the properties of the system under small moment of inertia and special descending commands are investigated, some mathematical and simulation results are presented.

In following contents, part 2 is a concise review of the λ-model of human motor system, which is an integrated model based on the equilibrium point hypothesis [1]. The equations of the λ-model are listed. Part 3 is dealing with the stabilities of the λ-model under different situations, it suggested that the λ-model is stable and has a unique equilibrium point under certain conditions. Part 4 gives detailed investigation about the properties of the λ-model when the moment of inertia is tiny. At last, some simulation results are presented.

II. MODEL DESCRIPTION

The λ-model is one of the most popular human motor system models, which suggests that a fixed voluntary motor command can be described as a set of equilibrium positions called an invariant characteristic. Therefore, if we change external torque, the limb position will shift along the invariant characteristic until equilibrium was once again established [2]. For simplicity, we selected the simple versions from various literatures [3,4] for simulation.

It is assumed that the musculoskeletal system is combined with antagonists around a hinge joint, and the muscle activation may according with the changes in threshold angles of flexor (λ_f) and extensor (λ_e), which is determined by the non-negative reciprocal (R) and coactivation (C) commands. The dynamic threshold angles (λ_f^*, λ_e^*) depend on the static counterpoints and angular velocity ($\omega = d\theta / dt$):

$$\lambda_f = R - C, \lambda_e = R + C, \lambda_f^* = \lambda_f - \mu\omega + \rho_f, \lambda_e^* = \lambda_e - \mu\omega - \rho_e \qquad (1)$$

where μ is a damping factor [5]. The changes of the threshold angles ($\rho_f,\ \rho_e$) in equation (1) due to reciprocal inhibition between antagonist motor neurons adjusted by muscle afferents and segmental inter neurons. It is assumed that these inter neurons just begin their recruitment when the threshold is less than that for motor neurons of homonymous muscles, otherwise, the behavior of the inter neurons closely resembles that of motor neurons [3]:

$$\rho_f = r[\lambda_e - \mu\omega + h - \theta]^+, \rho_e = r[\theta - \lambda_f + \mu\omega + h]^+ \qquad (2)$$

where r represents the strength of reciprocal inhibition, $[x]^+ = x$ if $x \geq 0$ and $[x]^+ = 0$ if $x < 0$.

Then, muscle activation arises when the joint angle (θ) reaches the dynamic threshold angle, and the values are determined by a function of the difference between them. Thus, Muscle activation E_f and E_e can be calculated as following experimental equations:

$$E_f = e^{\alpha[\theta - \lambda_f^*]^+} - 1, E_e = e^{\alpha[\lambda_e^* - \theta]^+} - 1 \qquad (3)$$

For simplicity, torques produced by the flexor and extensor muscles were assumed to be proportional to E_f and E_e, respectively:

$$T_f = aE_f, Te = aE_e \qquad (4)$$

where a and α are constants. Equation (3) and (4) are consistent with experimental static torque and angle characteristics for constant value of λ [2]. Meanwhile, because E_f and E_e are function of velocity, they also include dynamic torques. The contribution of passive issues to elasticity and damping was ignored. The acceleration ($\varepsilon = d^2\theta / dt^2$) is calculated from the Newtonian equation of motion:

$$I\varepsilon = T_e - T_f \qquad (5)$$

where I is the experimentally measured moment of inertia.

III. STABILITY ANALYSIS

Given a human motor system model, the most important problem is stability, because an unstable motor system is useless and dangerous to human. As the λ-model is built based on the equilibrium point hypothesis, the determination of equilibrium point and stability are important. Here, we present new stability results of the model. The results are obtained based on the contraction theory presented in [6].

For the λ-model given in (1) – (5) in the previous part, it can be described in the state-space form [7]:

$$\begin{bmatrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x_1, x_2, t) \end{bmatrix}$$

where f is given by $f(x_1,x_2,t)=(T_e-T_f)/I$, x_1 and x_2 are the state variables, corresponding to $x_1=\theta$, and $x_2=\omega$, where ω represents the angular velocity. Note that $f(x_1,x_2,t)$ is a nonlinear piecewise function.
Then, we have following results:

Theorem 1:
If

1) $h \ge 0$; $r > 0$; $R \ge 0$; $C \ge 0$;

2) $\lambda_f \ge h$.

it has

$$\begin{cases} \lambda_e + h \ge \dfrac{\lambda_f + r\lambda_e + rh}{1+r} \ge \lambda_f - h \\ \lambda_e + h \ge \dfrac{\lambda_e + r\lambda_f - rh}{1+r} \ge \lambda_f - h \end{cases}$$

From theorem 1, the possible operation regions of the λ-model can be summarized to 6 cases. Consider the stability of system in these cases, it gives theorem 2:

Theorem 2: *For the λ-mode given in equations (1)-(5) with the conditions,*

1) $h \ge 0$; $0 < r < 1$; $R \ge 0$; $C \ge 0$; $\lambda_f \ge h$; $\mu > 0$; $I > 0$;

2) the product $a \times \alpha > 0$.

it has:

for $C \ge \dfrac{rh}{1-r}$, *there is only one equilibrium point and it is given by* $\begin{cases} x_1(\infty) = \dfrac{\lambda_f + \lambda_e}{2} = R \\ x_2(\infty) = 0 \end{cases}$;

for $C < \dfrac{rh}{1-r}$, *the system will not have a unique equilibrium point and may oscillate.*

IV. FURTHER INVESTIGATION

Theorem 2 shows if $C < \dfrac{rh}{1-r}$, the system may trap in oscillation, but the question is whether the system should always oscillate under this condition? In this part, some investigations and simulation results are presented to provide a preliminary solution of this problem.
At first, we will show the state-space equations of the system when $C < \dfrac{rh}{1-r}$. If $C < \dfrac{rh}{1-r}$, it has

$$C < \frac{rh}{1-r} \Rightarrow \frac{\lambda_e + r\lambda_f - rh}{1+r} < \frac{\lambda_f + r\lambda_e + rh}{1+r}$$

Then, the operation regions can be combined into the following 5 cases:

(1) When $\dfrac{\lambda_f + r\lambda_e + rh}{1+r} \le \theta + \mu\omega \le \lambda_e + h$, then the state-space equation becomes

$$\begin{bmatrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{a}{I}[1 - e^{\alpha(x_1 - \lambda_f + \mu x_2 - r\lambda_e + r\mu x_2 - rh + rx_1)}] \end{bmatrix} \tag{6}$$

Thus, its equilibrium point is determined as

$$\begin{cases} x_1(\infty) = \dfrac{\lambda_f + r\lambda_e + rh}{1+r}, \\ x_2(\infty) = 0. \end{cases}$$

We have the Jacobian matrix of

$$\begin{bmatrix} 0 & 1 \\ a_{21} & a_{22} \end{bmatrix}$$

where

$$a_{21} = -\frac{a}{I}\alpha(1+r)[e^{\alpha(x_1 - \lambda_f + \mu x_2 - r\lambda_e + r\mu x_2 - rh + rx_1)}]$$

$$a_{22} = -\frac{a}{I}\alpha(\mu + r\mu)[e^{\alpha(x_1 - \lambda_f + \mu x_2 - r\lambda_e + r\mu x_2 - rh + rx_1)}]$$

(2) When $\lambda_f - h \le \theta + \mu\omega \le \dfrac{\lambda_e + r\lambda_f - rh}{1+r}$, then the state-space equation becomes

$$\begin{bmatrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = \varepsilon = \dfrac{a}{I}[e^{\alpha(\lambda_e - \mu x_2 - rx_1 + r\lambda_f - r\mu x_2 - rh - x_1)} - 1] \end{bmatrix} \tag{7}$$

and its equilibrium point is given by

$$\begin{cases} x_1(\infty) = \dfrac{\lambda_e + r\lambda_f - rh}{1+r}, \\ x_2(\infty) = 0 \end{cases}$$

In this case, the Jacobian matrix has its elements

$$a_{21} = -\frac{a}{I}\alpha(1+r)[e^{\alpha(\lambda_e - \mu x_2 - rx_1 + r\lambda_f - r\mu x_2 - rh - x_1)}]$$

$$a_{22} = -\frac{a}{I}\alpha(\mu + r\mu)[e^{\alpha(\lambda_e - \mu x_2 - rx_1 + r\lambda_f - r\mu x_2 - rh - x_1)}]$$

(3) When $\theta + \mu\omega > \lambda_e + h$, then, the state-space equation becomes

$$\begin{bmatrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = \varepsilon = \dfrac{a}{I}[1 - e^{\alpha(x_1 - \lambda_f + \mu x_2)}] \end{bmatrix} \tag{8}$$

In this case, the equilibrium point is

$$\begin{cases} x_1(\infty) = \lambda_f \\ x_2(\infty) = 0 \end{cases}$$

and the Jacobian matrix has the form

$$\begin{bmatrix} 0 & 1 \\ \dfrac{a}{I}\alpha[-e^{\alpha(x_1 - \lambda_f + \mu x_2)}] & \dfrac{a}{I}\alpha\mu[-e^{\alpha(x_1 - \lambda_f + \mu x_2)}] \end{bmatrix}$$

(4) When $\theta + \mu\omega < \lambda_f - h$, then the state-space equation becomes

$$\begin{bmatrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = \varepsilon = \dfrac{a}{I}[e^{\alpha(\lambda_e - \mu x_2 - x_1)} - 1] \end{bmatrix} \tag{9}$$

In this case, its equilibrium point is determined as

$$\begin{cases} x_1(\infty) = \lambda_e \\ x_2(\infty) = 0 \end{cases}$$

and the Jacobian matrix has the form

$$\begin{bmatrix} 0 & 1 \\ -\dfrac{a}{I}\alpha[e^{\alpha(\lambda_e - \mu x_2 - x_1)}] & -\dfrac{a}{I}\alpha\mu[e^{\alpha(\lambda_e - \mu x_2 - x_1)}] \end{bmatrix}$$

(5) When $\dfrac{\lambda_e + r\lambda_f - rh}{1+r} < \theta + \mu\omega < \dfrac{\lambda_f + r\lambda_e + rh}{1+r}$, then, the state-space equation becomes

$$\begin{bmatrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = \varepsilon = 0 \end{bmatrix} \tag{10}$$

and the Jacobian matrix has the form

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then we get the operation regions in θ-ω space as shown in Fig. 1. Note that the system states with the initial points given in items (1)-(4) may enter the adjacent regions as their equilibrium points are in the adjacent regions or on borders. But the middle region in item (5) is an unstable region. Thus, if the system converges monotonically to the equilibrium points, then it may have two equilibrium points as in items (1) and (2). If the system converges to the equilibrium points with oscillatory behavior, then it may enter the middle region. In this case, it will cross over the middle region and enter the next region opposite to the previous region. Therefore, the system may oscillate.

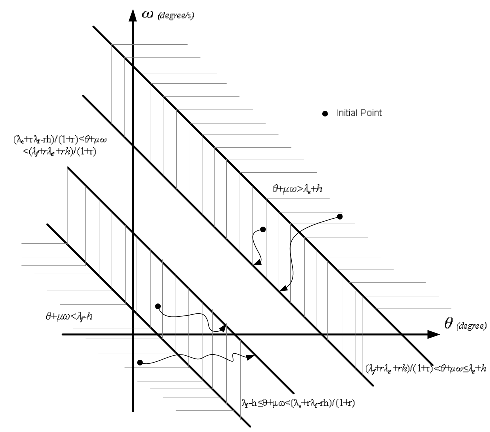

Fig.1: Operation regions when $C < \dfrac{rh}{1-r}$.

Therefore, the problem at the start of this part becomes: how can we judge whether the system will converge monotonically to the equilibrium point or not. Because the system states with initial points given in items (3) and (4) must enter the adjacent regions, we will only consider the regions defined in items (1) and (2), that is, the regions border upon middle region.

As mentioned before, the general format of the λ-model's Jacobian matrix is $\begin{bmatrix} 0 & 1 \\ a_{21} & a_{22} \end{bmatrix}$, therefore, the eigenvalues are:

$$\lambda_{\pm} = \frac{1}{2}[a_{22} \pm \sqrt{4a_{21} + a_{22}^{\ 2}}]$$

According to the knowledge of Jacobian matrix, if the eigenvalues have the imaginary part, the system will trap in oscillation. Therefore, it gives

$$4a_{21} + a_{22}^{\ 2} < 0$$

if the system converge to the equilibrium point with oscillatory behavior.

Apparently, in items 2 and 3, it has $a_{22}=\mu a_{21}$ and $a_{21}<0$. Then we have:

$$4a_{21} + a_{22}^{\ 2} = (4 + \mu^2 a_{21})a_{21} < 0 \Rightarrow 4 + \mu^2 a_{21} > 0 \Rightarrow \mu^2 a_{21} > -4$$

Combined with expressions of items (1) and (2), the above inequation becomes:

item (1):

$$\mu^2 a_{21} = -\frac{a\mu^2}{I}\alpha(1+r)[e^{\alpha(x_1 - \lambda_f + \mu x_2 - r\lambda_e + r\mu x_2 - rh + rx_1)}] > -4$$

$$\Rightarrow \frac{a\mu^2}{I}\alpha(1+r)[e^{\alpha(x_1 - \lambda_f + \mu x_2 - r\lambda_e + r\mu x_2 - rh + rx_1)}] < 4$$

item (2):

$$\mu^2 a_{21} = -\frac{a\mu^2}{I}\alpha(1+r)[e^{\alpha(\lambda_e - \mu x_2 - rx_1 + r\lambda_f - r\mu x_2 - rh - x_1)}] > -4$$

$$\Rightarrow \frac{a\mu^2}{I}\alpha(1+r)[e^{\alpha(\lambda_e - \mu x_2 - rx_1 + r\lambda_f - r\mu x_2 - rh - x_1)}] < 4$$

Therefore, if the moment of inertia is large enough, these inequations will be proved, that is, the system states with initial points given in items (1)-(4) will oscillate around middle region; otherwise, the system will converge monotonically to the equilibrium points on the border of middle region (λ_f and λ_e). This result suggests that if the moment of inertia of human limb is too small, the limb movement will stop stably at an equilibrium position, which is determined by the initial states.

IV. SIMULATION RESULTS

For verifying the above mathematical conclusions, simulation is carried out.

Model is used for simulation with following parameters [3]:

R=60°, C=1°, r=0.2,

I=0.115 kg•m^2, a=1.2 N•m, α=0.05 deg^{-1}, h=10°, μ=0.58 s.

Then, the threshold angles of flexor (λ_f) and extensor (λ_e) can be calculated as:

$\lambda_f = R - C = 59°$, $\lambda_e = R + C = 61°$,

and $\dfrac{rh}{1-r} = \dfrac{0.2*10}{0.8} = 2.5° > C$

At first, the performance of this model under different moments of inertia is investigated. Fig. 2 shows the system dynamic responses with different initial values and moments of inertia. The blue curve and red curve start from same

initial point with the different moments of inertia. The result shows that the blue curve oscillates around the middle region, but red curve ceases as a horizontal line, which is also the upper border of middle region. It suggests that the movement of the limb with small moment of inertia will stop at a certain position, even under the impotent descending commands. Note the pink curve and red curve start from different initial points with the same moment of inertia. These two curves converge to different horizontal lines. Because the system enter the middle region with different velocity, and the acceleration of the middle region is zero, that is, the system will keep its own velocity before entering the next region, and will stop at the border of next region by the enormous deceleration.

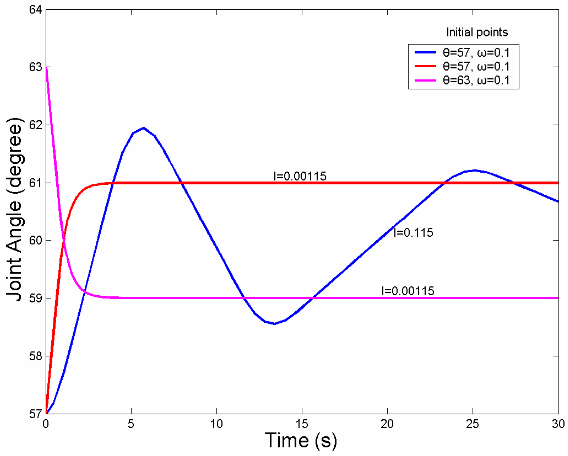

Fig. 2. The dynamic response of the system with different initial points (θ, ω) and moment of inertia (*I*)

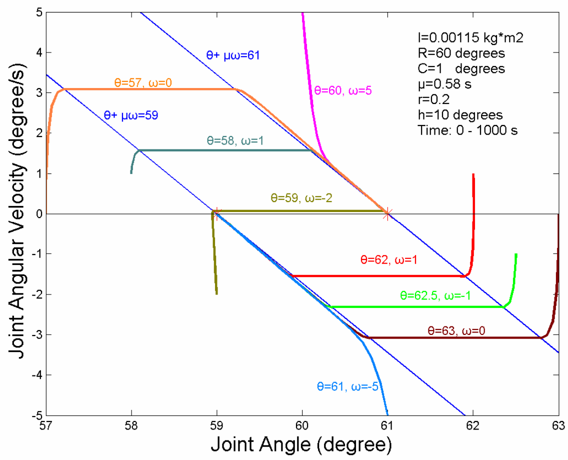

Fig. 3. The dynamic response of system with different initial points (θ, ω) and the tiny moment of inertia.

Then we face a new problem: when the system will enter the middle region and converge to the border of opposite region, and when the system will monotonically converge to the equlibrium point at the border without entering the middle region. Fig 3 shows the dynamic response of the system with different initial points and the tiny moment of inertia. It suggests that the system states near the two candidate equilibrium points determine the final equilibrium point. If the system tends to the equilibrium point of current region before entering the middle region, and is decelerated when it is close to this equilibrium point, the system will not span the middle region. Otherwise, the system will converge to the equilibrium point in opposite region.

V. Conclusion

Human motor performance appears to be remarkably flexible and dexterous; this requires the model of human motor system must be stable and easy to control. Therefore, the discussion for properties of the model under different conditions is very necessary and significant. In this short report, an important human motor system model - λ-model is described and investigated. The results about stability of this model are presented, which suggest that the λ-model will be stable and have a unique equilibrium point under certain conditions. Especially, when the moment of inertia of limb is tiny, the stability and final equilibrium point will be determined by the values of moments of inertia and initial points. These results should be served as a guideline on the forecast for the behavior of some special motor systems, and may be useful for studying the stability of other nonlinear piecewise functions.

References

[1] A. G. Feldman, "Functional tuning of the nervous system with control of movement or maintenance of a steady posture. III. Mechanographic analysis of execution by man of the simplest motor tasks". Biophysics, vol. 11, pp. 766–775, 1966.

[2] Feldman AG. "Once more on the equilibrium-point hypothesis (λ model) for motor control". J Mot Behavior vol. 18, pp.17-54. 1986.

[3] St-Onge, N., H. Qi, and A.G. Feldman, "The patterns of control signal underlying elbow joint movements in humans". Neuroscience Letters, vol 164, pp. 171-174. 1993.

[4] Reza Shadmehr, "The Equilibrium Point Hypothesis for Control of Movements". 1998.

[5] Feldman, A.G., et al., "The origin of electromyograms – Explanations based on the equilbrium point hypothesis, in Multiple Muscle Systems: Biomechanics and Movement Organization", J. Winters and S.L.-Y. Woo, Springer-Verlag New York Inc.: New York, NY. pp. 195-213. 1990.

[6] Winfried Lohmiller and Jean-Jacques E. Slotine, "On contraction analysis for non-linear systems" Automatica Vol.34, No.6, pp. 683-696, 1998.

[7] Jean-Jacques E. Slotine, Weiping Li, "Applied Nonlinear Control", Prentice-Hall International, Inc.1991.